Materials Efficiency of Cable-Stayed Bridges versus Suspension Bridges

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Abstract

Which type of bridge uses, in principle, the greater amount of steel cable: the classic suspension bridge, or the more recent harp-design cable-stayed bridge (all other factors being held constant)? Do the two designs have the same scaling laws? If yes, is one or the other better by a constant factor? Answers: Same scaling. Cable-stayed wins by a factor 3/2. But those "other factors" (here ignored) likely dominate real designs.

1 Introduction

Writing before the turn of the 21st century, engineer-author Henry Petroski (1942–2023) [Pet93] cautioned against an already noticeable trend in long-span bridge design, that harp-design cable-stayed bridges (Figure 1) were overtaking the classic suspension bridge design (Figure 2). Petroski thought that bridge design should be—remain—more conservative. He noted that more than fifty years elapsed from the 1883 inauguration of the Brooklyn Bridge, the first steel-cable suspension bridge, until a particular design failure mode of that basic design manifested itself catastrophically in the Tacoma Narrows Bridge failure of 1948. Petroski analogously predicted a design failure in some cable-stayed bridge within thirty years (twenty having already elapsed). That hasn't yet happened. Instead, cable-stayed bridges of canonical harp design have increasingly become familiar features in the built environment.

For any particular bridge, the choice of design is influenced by multiple factors, including site geometry and geology, materials cost and availability, construction costs and risks, aesthetics, and more. One reads that cable-stayed bridges make use of more modern steel alloys—but so could new suspension bridge designs. It is said that cable-stayed bridges are easier to construct; this may be so.

Still, one might wonder about the basic physics (or statics) of the problem. One can imagine (Figure 3) two bridges with identical spans, identical load-bearing towers, and identical decks, road beds and loads. The bridges differ only in how the cables are deployed, whether as a main cable plus suspender cables (suspension), or as a harp (cable-stayed). Which uses more steel cable of identical working strength?

Figure 1: This portion of the Weinan Weihe Grand Bridge exemplifies the cable-stayed harp design. [Credit: chinanews.com]

Figure 2: The Golden Gate Bridge in San Francisco exemplifies the classic suspension bridge design. [Credit: https://stock.adobe.com/au/499521989]

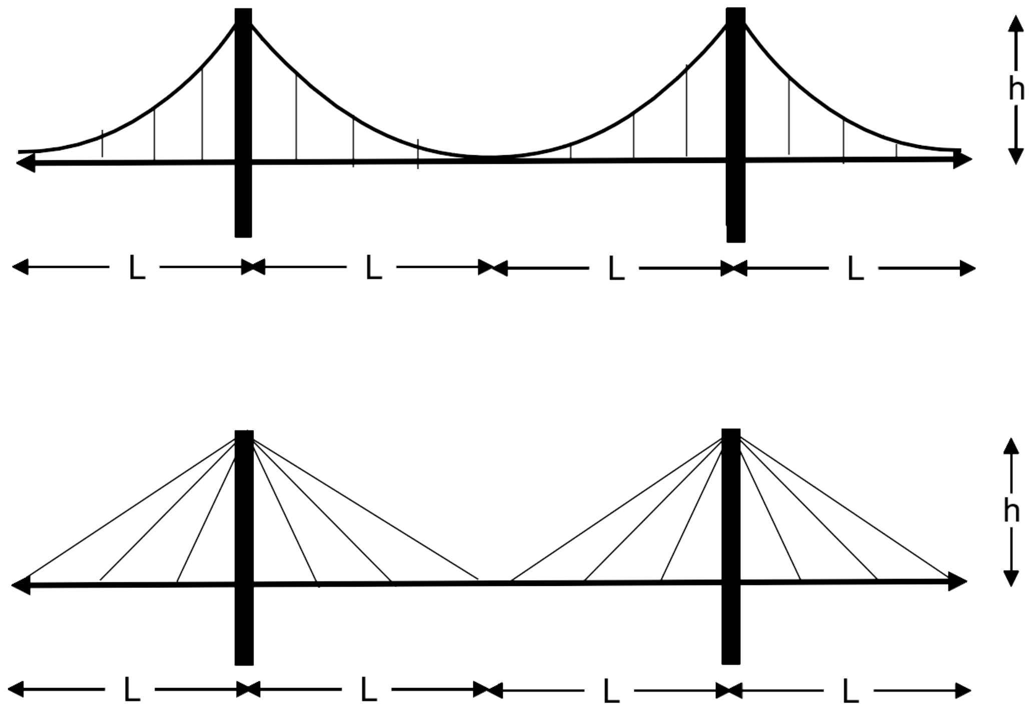

Figure 3: Suspension and cable-stayed bridges of identical geometry, differing only in their use of supporting cable. Which uses less steel?

That is the question this note answers in the highly idealized case. Comparing the top and bottom drawings in Figure 3, we see that the two designs carry load from the same points on the deck to, ultimately, the same points on the towers, differing only geometrically by the paths and angles that transmit that load. So, intuitively, we might expect that their use of cable differs only by a geometrical factor of order unity. That intuition will turn out to be basically correct. We will learn the value of that factor, and its dependence on the dimensionless ratio h/L.

1.1 Figure of Merit

We use classic units. High-carbon steel, as might be used in ideal bridge cables, can have a tensile strength of 300,000 lbs/in^2 and a working strength, denoted σ, of one-tenth that. Its typical density, denoted ρ, is 0.284 lbs/in^3.

The ratio σ/ρ, with the dimensions of a length, is the length of a vertical hanging cable of uniform cross-section whose tension (at the top) first exceeds its working strength. For the values given, this is $\approx$ 8800 ft., much longer than the semi-span L of

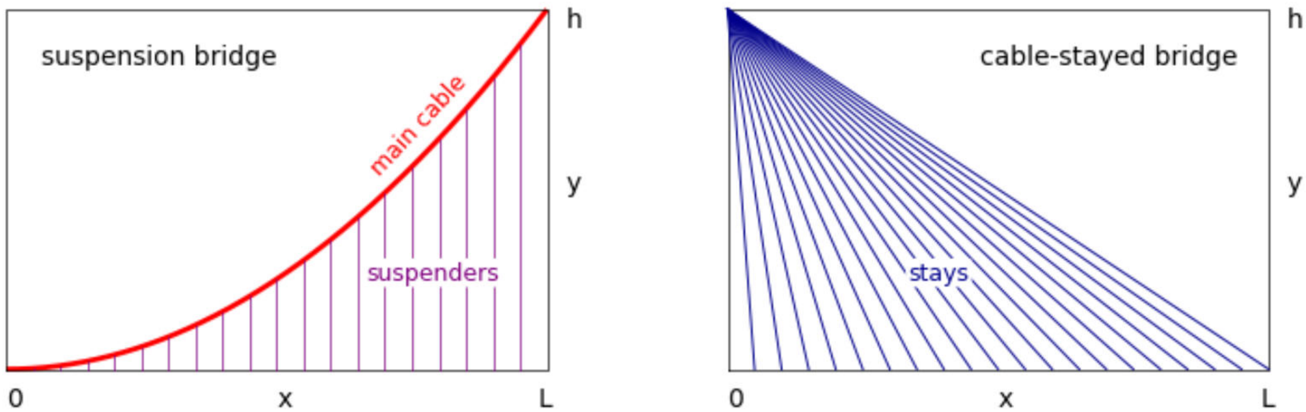

Figure 4: Coordinates and terminology used in the analysis.

the bridges in Figure 3 that we intend to design. That is, we assume

$$L \ll \sigma/\rho \tag{1}$$

and we can then neglect the self-load of the cables.

We will also assume no engineering barriers to tapering cables along their length so that they are at all points bearing a stress σ, equal to their working (not tensile!) strength. This maximally economizes, with no unnecessary steel.

Mass-per-length and cross-sectional area A of a cable under tension (force) T are thus related by

$$T = \sigma A, \qquad dM = (\rho A)\, d\ell = \frac{\rho}{\sigma} T\, d\ell \tag{2}$$

It is thus convenient to define as a figure of merit an effective mass $\widehat{M}$,

$$\widehat{M} = \frac{\sigma}{\rho} M = \int T(\ell)\, d\ell \tag{3}$$

where $T(\ell)$ is the tension (force) as it varies along a cable length coordinate ℓ. Minimizing $\widehat{M}$ is equivalent to minimizing the mass M, which can then be calculated multiplying by ρ/σ.

2 Suspension Bridge

Figure 4 (left) shows coordinates for a single semi-span of the bridge. Suppose that the total weight of the deck is W, uniformly distributed as w per unit length. We will calculate separately the effective mass of the main cable, $\widehat{M}_{\text{main}}$, and the effective mass of the suspenders $\widehat{M}_{\text{susp}}$ and then add them.

2.1 Main Cable

We will need to calculate the shape of the main cable, $y(x)$, and the vector tension $\mathbf{T}$ along it, with (x, y) components (T_x, T_y). We can immediately deduce these conditions,

$$y(0) = 0 \qquad \text{(don't make the suspenders longer than need to be)} \tag{4}$$
$$y'(0) = 0 \qquad \text{(left-right symmetry)} \tag{5}$$
$$y(L) = h \qquad \text{(boundary condition)} \tag{6}$$

prime denoting derivative w.r.t. x. Because the main cable is horizontal at $x = 0$ (equation (5)), we have

$$T_y(0) = 0 \tag{7}$$

By horizontal force balance on each element of the cable, we have

$$T_x(x) = \text{constant}. \tag{8}$$

Starting at $x = 0$ where $T_y = 0$, vertical tension increases as an increasing weight of deck is supported, so,

$$T_y' = w \quad \Longrightarrow \quad T_y(x) = wx, \quad T_y(L) = W \tag{9}$$

We could have written the final condition immediately, because the whole deck is supported at $x = L$ by the tower.

For the shape of the main cable, we have

$$y' = \frac{T_y}{T_x} = \frac{wx}{T_x} \Longrightarrow y(x) = \frac{1}{2}\frac{w}{T_x}x^2 \tag{10}$$

or using equation (6),

$$y(x) = h\left(\frac{x}{L}\right)^2, \quad y'(x) = \frac{2h}{L^2}x\,, \quad T_x = \frac{1}{2}\frac{wL^2}{h} = \frac{1}{2}\frac{WL}{h} \tag{11}$$

(That the shape of a perfectly uniformly loaded cable is exactly parabolic is a classic exercise in beginning engineering textbooks.)

The magnitude of total tension along the main cable is (using (10))

$$|\mathbf{T}(x)| = \sqrt{T_x^2 + T_y^2} = T_x\sqrt{1 + y'^2} \tag{12}$$

On the other hand, arc length along the main cable has the same functional form,

$$ds = \sqrt{1 + y'^2}\,dx \tag{13}$$

Our figure of merit $\widehat{M}_{\text{main}}$ is thus calculated as

$$\begin{aligned}\widehat{M}_{\text{main}} &= \int |T|\,ds = \int T\frac{ds}{dx}\,dx \\ &= \frac{1}{2}\frac{WL}{h}\int_0^L (1 + y'^2)\,dx \\ &= \frac{1}{2}\frac{WL^2}{h}\left[1 + \frac{4}{3}\left(\frac{h}{L}\right)^2\right]\end{aligned} \tag{14}$$

Although h/L is generally small, equation (14) is not a series expansion, but an "exact" result under the assumptions made. The leading term can be understood as (up to a constant) the weight W of the deck multiplied by a factor $L/h > 1$ due to the unfavorable (close to horizontal) direction of the cable, then multiplied by the length L in the definition of $\widehat{M}$.

2.2 Suspenders

The number of suspender cables is irrelevant if large, because each supports just its own section of deck, scaling accordingly. We can thus take the continuum limit,

$$\widehat{M}_{\text{susp}} = \int_0^L wy\,dx = \int_0^L wh\left(\frac{x}{L}\right)^2 dx = \frac{1}{3}hLw = \frac{1}{3}hW \tag{15}$$

Note that this is only of order the second (generally smaller) term in equation (14).

2.3 Total

Adding equations (14) and (15) gives the final result

$$\widehat{M} = W\left(\frac{L^2}{2h} + h\right) \tag{16}$$

3 Cable-Stayed Bridge

Referring to Figure 4 (right), and calculating in the continuum limit of many stays, consider the stay that attaches at position x. It supports a vertical deck weight $w\,dx$, but at an angle with tangent $h/\sqrt{x^2+h^2}$, giving it a tension of total magnitude

$$T = \frac{w\,dx}{h/\sqrt{x^2+h^2}} \tag{17}$$

and a length of $\ell = \sqrt{x^2+h^2}$. The figure of merit is thus

$$\widehat{M} = \int_0^L \ell\,T = \frac{w}{h}\int_0^L (x^2+h^2)\,dx = W\left(\frac{L^2}{3h} + h\right) \tag{18}$$

4 Comparison

One sees that equations (16) and (18) each have two terms, a first that dominates when $h \lesssim L$ (always true in practice) and a second that dominates in the unlikely limit of very tall towers and a narrow span. The first terms have the same scaling in W, L, and h, but differ by a geometrical factor $3/2$, favoring the cable-stayed design. The second terms are identical as hW, because in the limit of tall towers, the deck of weight W is supported by effectively vertical cables of length h.

No value of h/L favors the suspension bridge, so it is cable-stay by a nose, independent of whether the bridge is big or small, tall or short, supporting a load heavy or light.

Really, as mentioned from the start, there are many other engineering considerations in play. But, all else being equal, statics alone gives cable-stayed bridges the advantage in materials cost. Sorry about that, Mr. Petroski.

References

[Pet93] Henry Petroski. Predicting disaster. *American Scientist*, 81(2):110–113, 1993.